The Generalized Extreme Value (GEV) Distribution, Implied Tail Index and Option Pricing

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Abstract
Crisis events such as the 1987 stock market crash, the Asian Crisis and the collapse of Lehman Brothers have radically changed the view that extreme events in financial markets have negligible probability. This article argues that the use of the Generalized Extreme Value (GEV) distribution to model the implied Risk Neutral Density (RND) function provides a flexible framework that captures the negative skewness and excess kurtosis of returns, and also delivers the market implied tail index. We obtain an original analytical closed form solution for the Harrison and Pliska [1981] no arbitrage equilibrium price for the European option in the case of GEV asset returns. The GEV based option pricing model successfully removes the in-sample pricing bias of the Black-Scholes model, and also shows greater out of sample pricing accuracy, while requiring the estimation of only two parameters. We explain how the implied tail index is efficacious at modelling the fat tailed behaviour and negative skewness of the implied RND functions, particularly around crisis events.

The last two decades have been marked by crisis events in financial markets. These include the 1987 stock market crash, the Asian Crisis (July–October 1997), the September 1998 LTCM debacle, the bursting of the high technology Dot-Com bubble of 2000-02 with about 30% losses of equity values, events such as 9/11, sudden corporate collapses of the magnitude of Enron and Lehman Brothers, and most recently, the 2007/08 credit crisis which has been considered to be the greatest since the Great Depression. There has been a radical shift in the view held by policy makers, finance academics and practitioners who now feel that extreme events in financial markets cannot be ignored as outliers with negligible probability. In mainstream financial theory, extreme events which occur with small probabilities have not been a matter of concern as in the dominant model of lognormal asset prices the probability of extreme events such as the stock market crash of October 1987 is virtually non-existent.[1] There has been a growing pragmatic and theoretical shift in interest from the modelling of 'normal' asset market conditions to the shape and fatness of the tails of the distributions of asset returns which characterize statistical models for extreme events.

Extreme value theory is a robust framework to analyse the tail behaviour of distributions. Extreme value theory has been applied extensively in hydrology, climatology and also in the insurance industry. Embrechts et. al. [1997] is a comprehensive source on extreme value theory and applications. Despite early work by Mandelbrot [1963] on the possibility of fat tails in financial data and evidence on the inapplicability of the assumption of log normality in option pricing, a systematic study of extreme value theory for financial modelling and risk management has only begun recently.[2]

The objective of this article is to use the Generalized Extreme Value (GEV) distribution in the context of European option pricing with the view to overcoming the problems associated with existing option pricing models. Within the Harrison and Pliska [1981] asset pricing framework, the risk neutral probability density (RND) function exists under an assumption of no arbitrage. By definition of a no arbitrage equilibrium, the current price of an asset is the present discounted value of its expected future payoff given a risk-free interest rate where the expectation is evaluated by the RND function. Breeden and Litzenberger [1978] were first to show how the RND function can be extracted from traded option prices. The Black-Scholes [1973] and lognormal based RND models have well known drawbacks. First, the implied volatility smiles or smirks are inconsistent with the constancy required in the lognormal case for volatility across different strikes for options with the same maturity date. Further, this class of models cannot explicitly account for the negative skewness and the excess kurtosis of asset returns. Since, Jackwerth and Rubinstein [1996] demonstrated the discontinuity in the implied skewness and kurtosis across the divide of the 1987 stock market crash - a large literature has developed which aims to extract the RND function from traded option prices so that the skewness and fat tail properties of extreme market events are better captured than is the case in lognormal models.

Pricing biases caused by left skewness of asset returns that cannot be captured in the implied lognormal asset pricing models are now well understood (see,Corrado and Su [1996,1997], Savickas

[1] As noted by Jackwerth and Rubinstein [1996] in a lognormal model of assets prices, the market crash on 19 October 1987 with a 29% fall of S&P 500 futures prices has a probability of 10^{-160}, an event which is unlikely to happen even in the life time of the universe.

[2] Embrechts et. al. [1999], Mc Neil [1999] and Embrechts [2000] consider the potential and limitations of extreme value theory for risk management. Dowd [2002] gives a good account of these developments and a recent survey of extreme value theory for finance can be found in Rocco [2010].

[2002]). Typically, in periods when the left skewness of asset prices increases, the Black-Scholes model will overprice out-of-the-money call options and underprice in-the-money call options relative to when there is greater symmetry in the distribution function. This article shows how the option price is highly sensitive to changes in the tail shape, which is distinct to its sensitivity to the variance of the returns distribution. We find that the traded option price implied GEV model for the RND yields results that strongly challenge traditionally held views on tail behaviour of asset returns based on Gaussian distributions which predicate simultaneous existence of thin tails in both directions during all market conditions. The GEV distribution, which is governed by the tail shape parameter, is found to switch tail shape with underlying market conditions. During extreme market drawdowns, a positive value for the tail shape parameter results in significant skewness in the probability mass of the GEV density function for losses and implies extreme price drops with the large probability mass on the right and a truncated tail in the other direction, implying an upper bound on possible gains. To date, proposed option pricing models intended to deal with both the fat tail and the skew in asset returns have failed to highlight the above characteristic features of fat tailed distributions. They have also run into problems ranging from a lack of closed form solution, a large number of parameters needed or the lack of easy interpretation of implied parameters. These factors have prevented many of these models from being of practical use in pricing and hedging options or in risk management for extreme market conditions.

This article argues for the use of the Generalized Extreme Value (GEV) distribution for asset returns in an option pricing model for the following reasons:

(i) It can provide a closed form solution for the European option price.

(ii) It yields a parsimonious European option pricing model, with only two parameters to estimate, the tail shape parameter and the scale parameter.

(iii) It provides a flexible framework that subsumes as special cases a number of classes of distributions that have been assumed to date in more restrictive settings. The GEV distribution encompasses the three main classes of tail behaviour associated with the Fréchet type fat tailed distributions and the thin and short tailed Weibull and Gumbel classes.

(iv) When the GEV distribution is of Fréchet type, it exhibits a fat tail on the right and a truncated tail on the left. Since extreme economic losses are more probable than extreme economic gains, we adopt the Fréchet distribution to model extreme losses. To this end, we follow the practice of the insurance industry, Dowd [2002, p 272], and model returns as negative returns. As a result, when extreme events are prominent, the GEV model yields a Fréchet type implied density function for negative returns, signifying higher probabilities of price drops.

(v) Most significantly, the GEV option pricing model can deliver the market implied tail index for asset returns. It is important to capture market perception of fat tailed behaviour in asset returns in a manner which is interspersed with thin and short tailed Gumbel and Weibull values for the tail index which characterize more normal market conditions. Hence, the market implied tail index is found to be time varying in a way that mirrors the lack of invariance in the recursively estimated tail index of asset returns (see, Quintos, Fan and Phillips [2001]) with jumps in the fat tailedness in crisis periods.

(vi) We show how the GEV option pricing model removes the well known pricing biases associated with the Black-Scholes, by capturing the time varying levels of skewness and kurtosis. We also show how the GEV model yields superior pricing accuracy out of sample, as GEV implied RNDs are more capable of capturing extreme market conditions than other option pricing models.

(vii) Having obtained a closed form solution for the option pricing model, we can also obtain a closed form solution for the new "greek" in the lexicon of option pricing, which measures the sensitivity of the option price to the tail index.

(viii) The closed form delta hedging formulation can also be given.

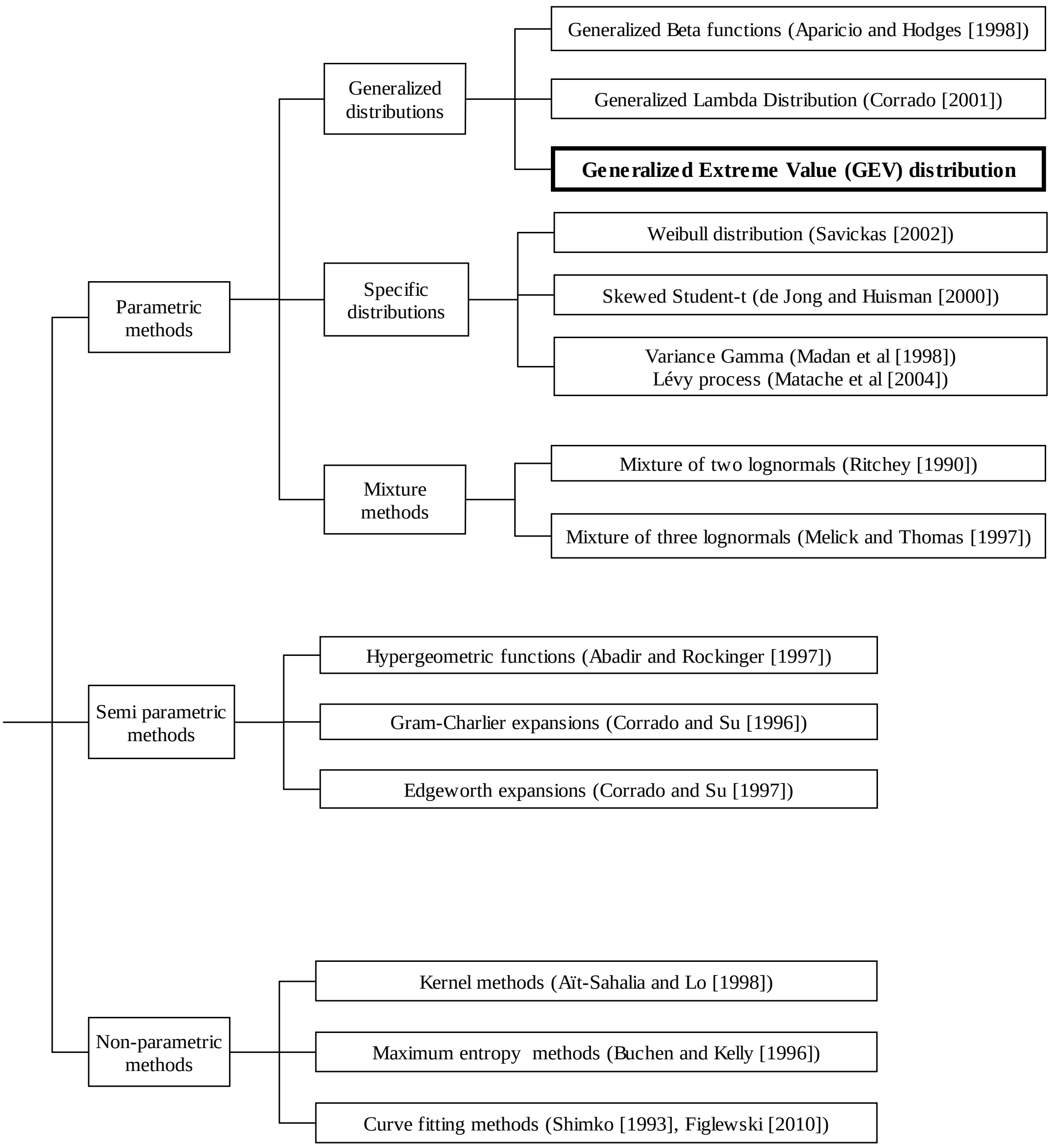
This article covers the first six features listed above of the GEV RND model of option pricing and we leave the last two for further work.

We will now briefly comment on how the GEV RND based option pricing model fits into the large edifice, given in Exhibit 1 below, built from the different methods used for the extraction of the implied distributions and their respective option pricing models that have arisen since the work of Breeden and Litzenberger [1978]. Based on Jackwerth [1999] survey, the different methods can be classified into three main categories: parametric, semi parametric and non-parametric. Parametric methods can be divided into three sub-categories: generalized distribution methods, specific distributions and mixture methods. Generalized distribution methods introduce more flexible distributions with additional parameters beyond the two parameters of the normal or lognormal distributions. Within this subcategory, Aparicio and Hodges [1998] use generalized Beta functions of the second kind, which are described by four parameters, and Corrado [2001] uses the generalized Lambda distribution. Under the specific distributions being assumed for the RND function, the Weibull distribution is used by Savickas [2002], and the skewed Student-t by de Jong and Huisman [2000]. The Variance Gamma distribution used by Madan, Carr and Chang [1998], and Levy processes used among others by Matache, Nitsche and Schwab [2004] are more recent specifications with these methods having parameters that can control fat tails and skewness of the asset price. Up to seven parameters are associated with these models.

Finally, the third sub-category within parametric methods is the mixture methods, which achieve greater flexibility by taking a weighted sum of simple distributions. The most popular method here is mixture of lognormals. Ritchey [1990] and Gemmill and Saflekos [2000] use two lognormals, and Melick and Thomas [1997] use three lognormals. One problem associated with the mixture of distributions is that the number of parameters is usually large, and thus they may overfit the data. For example, the mixture of two lognormals needs to estimate five parameters.

Under the category of semi parametric methods, the Hypergeometric function was used by Abadir and Rockinger [1997], and expansion methods such as the Gram-Charlier and Edgeworth expansions, respectively, were used by Corrado and Su [1996] and Corrado and Su [1997]. The non-parametric methods can be divided again into three groups: kernel methods, maximum-entropy methods, and curve fitting methods. Kernel methods, implemented in Ait-Sahalia and Lo [1998], are related to regressions since they try to fit a function to observed data, without specifying a parametric form. Second, the methods based on maximum-entropy used by Buchen and Kelly [1996] find a non-parametric probability distribution that tries to match the information content, while at the same time satisfying certain constraints, such as pricing observed options correctly. In the third group in this category, there are the curve fitting methods that try to fit the implied volatilities with some flexible function. The most popular of these is Shimko [1993] who introduced the concept of smoothed implied volatility smiles which involved fitting typically a cubic or low order polynomial spline to obtain the middle portion of the RND function. The tails of the RND function were modelled as log normal. This approach was improved by Bliss and Panigirtzoglou [2002] with the use of a "smoothing spline" whilst retaining log normal tails. Figlewski [2010] made an advance on this by appending tails from the GEV distribution which are able to reflect extreme market conditions.

Exhibit 1
Classification of most common RND estimation methods



The model presented in this article, as highlighted in Exhibit 1, falls in the general category of parametric models, and more specifically, within the sub-category of generalized distributions. In order to estimate tail behaviour at high confidence levels, such as 99%, many non-parametric methods for RND estimation fail to capture tail behaviour of the distributions because of sparse data for options traded at very high or very low strikes prices. Hence, parametric models have become unavoidable. This, however, replaces sampling error with model error. In the next section, we give a brief introduction to Extreme Value Theory and present the Generalized Extreme Value (GEV) distribution and its properties to indicate how the flexibility of this three parameter class of distributions can capture skew and fat tails as and when dictated by the data with no *a priori* restrictions on the class of distribution. This data driven selection of the tail index mitigates model error.

The rest of the article is organized as follows. We develop the GEV option pricing model and the closed form solutions for the arbitrage free European call and put option prices are derived for the GEV based RND function. We then proceed to discuss the components of the closed form solution and their theoretical properties in terms of moneyness and then tail shape parameter. The empirical section reports on the results for the estimated implied GEV RND function and for its parameters based on the FTSE 100 European option price data from 1997 to 2009. The in sample fit of the postulated GEV option pricing model is compared with the benchmark Black-Scholes one and is found to be superior at all levels of moneyness and at all time horizons, removing the well known price bias of the Black-Scholes model. Out of sample pricing tests show that the GEV provides superior pricing performance compared to Black-Scholes, for one day ahead forecasts. The analysis of the time series characteristics of the implied tail index is given and the role of implied RND functions in event studies surrounding periods of "extreme" price falls of the FTSE-100 index is also discussed. Finally, we make concluding remarks and discuss future work.

EXTREME VALUE THEORY AND THE GEV DISTRIBUTION

Unlike the normal distribution that arises from the use of the central limit theorem on sample averages, the extreme value distribution arises from the limit theorem of Fisher and Tippet [1928] on extreme values or maxima in sample data. The class of GEV distributions is very flexible with the tail shape parameter ξ (and hence the tail index defined as $\alpha = \xi^{-1}$) controlling the shape and size of the tails of the three different families of distributions subsumed under it. These three families of distributions can be nested into a single parametric representation, as shown by Jenkinson [1955] and von Mises [1936]. This representation is known as the "Generalized Extreme Value" (GEV) distribution and is given by:

$$F_{\xi}(x) = \exp\left(-\left(1+\xi x\right)^{-1/\xi}\right) \qquad \text{with} \qquad 1+\xi x > 0, \quad \xi \neq 0 \tag{1.a}$$

Applying the formula that $(1+\xi x)^{-1/\xi} \to e^{-x}$, as $\xi \to 0$ we have:

$$F_0(x) = \exp(-e^{-x}) \tag{1.b}$$

The standardized GEV distribution, in the form in von Mises [1936] (see, Reiss and Thomas [2001], p. 16-17), incorporates a location parameter μ and a scale parameter σ, in addition to the tail shape parameter, ξ, and is given by:

$$F_{\xi,\mu,\sigma}(x) = \exp\left(-\left(1+\xi\frac{(x-\mu)}{\sigma}\right)^{-1/\xi}\right) \quad \text{with} \quad 1+\xi\frac{(x-\mu)}{\sigma} > 0 \quad \xi \neq 0 \qquad (2.a)$$

and

$$F_{0,\mu,\sigma}(x) = \exp(-e^{-\frac{(x-\mu)}{\sigma}}) \qquad \text{with} \qquad \xi = 0 \qquad (2.b)$$

The corresponding probability density functions obtained by taking the derivative of the distribution functions, are respectively:

$$f_{\xi,\mu,\sigma}(x) = \frac{1}{\sigma}\left(1+\xi\frac{(x-\mu)}{\sigma}\right)^{-1-1/\xi} \exp\left(-\left(1+\xi\frac{(x-\mu)}{\sigma}\right)^{-1/\xi}\right) \qquad \xi \neq 0 \quad (3.a)$$

and

$$f_{0,\mu,\sigma}(x) = \frac{1}{\sigma} e^{-(x-\mu)/\sigma} \exp(-e^{-(x-\mu)/\sigma}) \qquad \xi = 0 \quad (3.b)$$

We will now discuss how the tail shape parameter, ξ, determines both the higher moments of the density function and also the skew in the probability mass leading to truncation points in the distribution. The tail shape parameter ξ=0 yields thin tailed distributions with the tail index $\alpha = \xi^{-1}$ being equal to infinity, implying that all moments of the distribution are either finite or zero.[3] When ξ = 0, the GEV distribution belongs to the Gumbel class and includes the normal, exponential, gamma and lognormal distributions, where only the lognormal distribution has a moderately heavy tail. The Gumbel class has zero skew in the probability mass and displays symmetry in the right and left tails. Further, as seen in equation (2.b) there are no conditions truncating the distribution in either direction for values of x. The distributions associated with ξ > 0 are called Fréchet and these include well known fat tailed distributions such as the Pareto, Cauchy and Student-t distributions. Finally, in the case where ξ < 0, the distribution class is Weibull.[4] These are short tailed distributions with finite upper bounds and include distributions such as uniform and beta distributions. In distributions for which ξ ≠ 0**,** the equality condition in equation (2.a) imposes a truncation of the probability mass and a distinct asymmetry in the right and left tails such that when the probability mass is high at one tail signifying non-negligible probability of an extreme event in that direction, there is an absolute maxima (or minima) in the other direction beyond which values of x have zero probability. As shown in Reiss and Thomas [2001], kurtosis of the Fréchet distribution becomes infinite at ξ ≥ 0.25 (the tail index, α ≤ 4), and all higher moments including kurtosis and the right skew become infinite at ξ ≥ 0.33 (the tail index, α ≤ 3). Even for small positive values of ξ, approximately at about ξ = 0.1, the rate of growth of

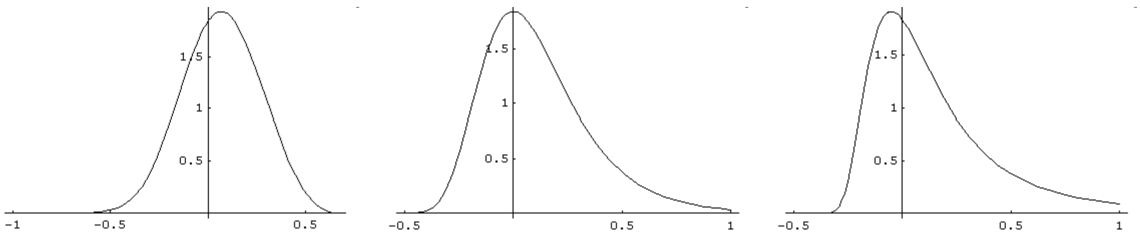
[3] The general rule is that the n^{th} and higher moments fail to be finitely integrable if the tail index is smaller than *n*. When ξ ≤ 0, all moments are finite or zero. However, when ξ≥ 0, only moments up to the integer part of the tail index, α =1/ ξ, exist with all other moments being infinite.
[4] Here we make reference to the Weibull distribution as defined in the context of extreme value theory, Embrechts [1997, p154].

skewness and kurtosis of the distribution, with both fast approaching infinite growth, results in a concentration of the probability density of the Fréchet distribution at the right tail. Thus, as ξ increases with $\xi > 0$, the truncation points at the left tail at which there is zero probability become more stringent. Note for $\xi < 0$, for the Weibull class of distributions, there is increased probability mass on the left tail and a truncation point given by the inequality in equation (2.a) at the right tail. However, it is well known (see, Reiss and Thomas [2001]) that at about $\xi = -0.3$, the Weibull distribution has zero skew and is indistinguishable from a Gumbel distribution.

As the probability of extreme economic losses are more likely than extreme gains, economic losses are modelled as a Fréchet distribution with high probability mass on the right tail. Exhibit 2a below illustrates the GEV density functions for negative asset returns for each of the three classes of distributions that the GEV can take based on the shape parameter ξ. Note, that the three graphs only differ in the value of ξ (the values considered for ξ are 0.3, 0, -0.3), having the same value for location (μ=0) and scale (σ=0.2) parameters. The initial stock price is assumed to be 100. The corresponding density functions for the price in each of the three cases for the tail shape parameter are shown in Exhibit 3. Note, the left skew in the price density function is greatest in Exhibit 3c, for the case when $\xi > 0$, and the negative returns density function belongs to the GEV-Fréchet class. Given the value being assumed, $\xi = 0.3$, in Exhibits 2c and 3c, as noted above, there is infinite kurtosis and a very stringent truncation on positive returns exceeding 0.667 or for the prices to rise above 166.67.[5] Likewise, for $\xi = -0.3$ in (2.a,3.a), we have zero probability for negative returns to be greater than 0.667 and the price to fall below 33.33. These upper and lower bounds on returns and prices implied by the GEV distribution play an important role in the analysis that follows. As will be shown later, for the range of values for the implied tail shape parameters for 30 day returns on the FTSE-100 index that we extract on a daily basis from option prices over the sample period from 1997-2009, the maximum value we obtain for ξ is +0.12. As this implies a tail index value of $\alpha = 8.33$, it is clear that this guarantees finite skewness and kurtosis for the risk neutral density function for the entire sample period. Further, on using equation (2.a), precise truncations values under the RND Q- measure can be determined for the levels of the stock index and for the returns on it. In the context of the option pricing model it is important to verify that the truncated values implied by a GEV based RND, ie. Q-impossible events are also not P-possible in terms of the empirically realized values for prices and returns.[6] This will be investigated in the next section.

Exhibit 2
Density functions for negative returns

[5] By rearranging the inequality in equation (2.a) and using the values being assumed for the GEV parameters, the truncation values denoted by x* for negative returns in Exhibits (2.c, 3.c)) and (2.a,3.a) are determined from $x^* > \mu - \sigma/\xi$.
[6] We are grateful to Stephen Figlewski for bringing this to our attention.

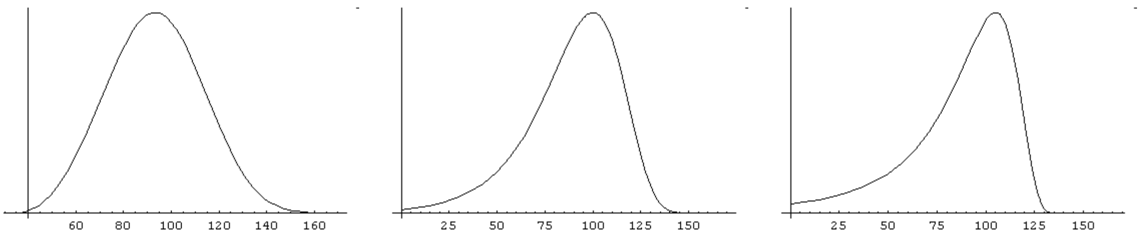


(a) ξ = - 0.3 (b) ξ = 0 (c) ξ = + 0.3

Notes: Density function of negative returns as modeled by the (a)GEV-Weibull, (b)GEV-Gumbel and (c)GEV-Fréchet.

Exhibit 3

Density functions for prices



(a) ξ = - 0.3 (b) ξ = 0 (c) ξ = + 0.3

Notes: Corresponding density function for prices where negative returns have been modeled as (a) GEV- Weibull, (b) GEV- Gumbel and (c) GEV- Fréchet.

THE GEV OPTION PRICING MODEL

Arbitrage Free Option Pricing and the Risk Neutral Density

Let S_t denote the underlying asset price at time t. The European call option with price C_t is written on this asset with strike K and maturity T. We assume the interest rate r is constant. Following the Harrison and Pliska [1981] result on the arbitrage free European call option price, there exists a risk neutral density (RND) function, $g(S_T)$, such that the equilibrium call option price can be written as:

$$C_t(K) = e^{-r(T-t)} E_t^Q\left[\max(S_T - K, 0)\right] = e^{-r(T-t)} \int_K^{\infty} (S_T - K) g(S_T) dS_T \tag{4}$$

Here, Q is the risk neutral measure and $E_t^Q[\cdot]$ is the risk-neutral expectation operator conditional on information available at time t, $g(S_T)$ is the risk-neutral density function of the underlying at maturity. Similarly, the arbitrage free option pricing equation for a put option is given by:

$$P_t(K) = e^{-r(T-t)} E_t^Q\left[\max(K - S_T, 0)\right] = e^{-r(T-t)} \int_0^{K} (K - S_T) g(S_T) dS_T \tag{5}$$

In an arbitrage-free economy, the following martingale condition must also be satisfied:

$$S_t = e^{-r(T-t)} E_t^Q(S_T) \tag{6}$$

European Call and Put Option Price with GEV returns

We assume that the RND function $g(S_T)$ in (4) for a holding period equal to time to maturity of the option is represented by the GEV distribution. We derive closed form solutions for the call and put option pricing equations by analytically solving the integrals in (4) and (5). For the purpose of obtaining an analytic closed form solution, it was found necessary to define returns as simple returns.[7] We define simple negative returns as follows:

$$L_T = -R_T = -\frac{S_T - S_t}{S_t} = 1 - \frac{S_T}{S_t} \tag{7}$$

In keeping with the extreme value distribution modelling of economic losses, L_T is assumed to follow the GEV distribution given in (3.a), $\xi \neq 0$,and hence the density function for the negative returns is given by:

$$f(L_T) = \frac{1}{\sigma}\left(1 + \frac{\xi(L_T - \mu)}{\sigma}\right)^{-1-1/\xi} \exp\left(-\left(1 + \frac{\xi(L_T - \mu)}{\sigma}\right)^{-1/\xi}\right) \tag{8}$$

Note, the relationship between the density function for L_T in (8) and the RND function $g(S_T)$ in (4) for the underlying price S_T is given by the general formula:

$$g(S_T) = f(L_T)\left|\frac{\partial L_T}{\partial S_T}\right| = f(L_T)\frac{1}{S_t} \tag{9}$$

On substituting (8) into (9), we obtain the RND function of the underlying price in terms of the GEV density function as in equation (3.a) :

$$g(S_T) = \frac{1}{S_t \sigma}\left(1 + \frac{\xi(L_T - \mu)}{\sigma}\right)^{-1-1/\xi} \exp\left(-\left(1 + \frac{\xi(L_T - \mu)}{\sigma}\right)^{-1/\xi}\right) \tag{10}$$

with

$$1 + \frac{\xi}{\sigma}(L_T - \mu) = 1 + \frac{\xi}{\sigma}\left(1 - \frac{S_T}{S_t} - \mu\right) > 0 \tag{11}$$

We will first consider the case when $\xi>0$ and $0 < \xi < 1$.[8] As already discussed, in this case the negative returns distribution is Fréchet and this implies that the price RND function $g(S_T)$ in (10), in order to satisfy the condition in (11), is truncated on the right. Hence, the upper limit of integration for

[7] Note, simple returns can give rise to the theoretical possibility of negative stock prices when $\xi > 0$. However, for purposes of option pricing this does not pose a problem as for the call price the lower limit of integration K for the stock price in equation (4) is always positive and likewise for the put price the lower limit of integration for the stock price in equation (5) is zero. Additionally, numerical results (not reported in this article) show that the implied GEV parameters (ξ, σ) obtained when using simple returns are not statistically different to the ones obtained using log returns.

[8] The condition $0< \xi <1$ is necessary to rule out the case that the first moment for the stock price at maturity is infinite and the option value becomes infinite. In this article all cases of $\xi>0$ will be constrained in this way. The closed form solution for the call option for the case when $0< \xi <1$ is identical to the one obtained for the case when $\xi < 0$. This is also true for the closed form solution for put option prices. Appendix B derives the closed form solutions for the call and the put options in the case of $\xi = 0$.

the call option price in (4) becomes $S_t(1-\mu+\sigma/\xi)$.[9] Substituting $g(S_T)$ in (10) into the call price equation in (4), we have:

$$C_t(K)=e^{-r(T-t)}\int_K^{S_t(1-\mu+\sigma/\xi)}(S_T-K)\frac{1}{S_t\sigma}\left(1+\frac{\xi(L_T-\mu)}{\sigma}\right)^{-1-1/\xi}\exp\left(-\left(1+\frac{\xi(L_T-\mu)}{\sigma}\right)^{-1/\xi}\right)dS_T \quad (12)$$

Consider the change of variable:

$$y=1+\frac{\xi}{\sigma}(L_T-\mu)=1+\frac{\xi}{\sigma}\left(1-\frac{S_T}{S_t}-\mu\right) \quad (13)$$

Under this change of variable, the underlying price S_T and dS_T can be written in terms of y as follows:

$$S_T=S_t\left(1-\mu-\frac{\sigma}{\xi}(y-1)\right) \quad \text{and} \quad dS_T=-S_t\frac{\sigma}{\xi}\,dy \quad (14)$$

Also, the density function in (10) for the underlying price at maturity in terms of y becomes:

$$g(y)=\frac{1}{S_t\sigma}\left(y^{-1-1/\xi}\right)\exp\left(-y^{-1/\xi}\right) \quad (15)$$

Note that under the change of variable the lower limit of integration for the call option equation in (12) becomes:

$$H=1+\frac{\xi}{\sigma}\left(1-\frac{K}{S_t}-\mu\right) \quad (16)$$

The upper limit of integration in (12) becomes 0. Substituting for S_T and dS_T as defined in (14) into (12), and using the new limits of integration we have:

$$C_te^{r(T-t)}=\int_H^0\left(S_t\left(1-\mu-\frac{\sigma}{\xi}(y-1)\right)-K\right)\frac{1}{S_t\sigma}\left(y^{-1-1/\xi}\right)\exp\left(-y^{-1/\xi}\right)\left(-S_t\frac{\sigma}{\xi}\right)dy \quad (17)$$

Simplifying and rearranging (17) we have:

[9] On the other hand, when $\xi < 0$ the GEV density function for S_T is truncated on the left, and therefore, the lower limit of integration for the call option price in (4) becomes $max[K, S_t(1-\mu+\sigma/\xi)]$ and the upper limit remains ∞. However, the closed form solutions for the call option are identical for both cases when $\xi > 0$ and $\xi < 0$. This also holds for put option prices.

$$
\begin{aligned}
C_t e^{r(T-t)} &= -\frac{1}{\xi}\int_H^0 \left(S_t\left(1-\mu-\frac{\sigma}{\xi}(y-1)\right)-K\right)\left(y^{-1-1/\xi}\right)\exp\left(-y^{-1/\xi}\right)dy \\
&= \frac{1}{\xi}\left[\frac{S_t\sigma}{\xi}\int_H^0 y\left(y^{-1-1/\xi}\right)\exp\left(-y^{-1/\xi}\right)dy-\left(S_t\left(1-\mu+\frac{\sigma}{\xi}\right)-K\right)\int_H^0\left(y^{-1-1/\xi}\right)\exp\left(-y^{-1/\xi}\right)dy\right] \\
&= \frac{1}{\xi}\left[\frac{S_t\sigma}{\xi}\psi_1-\left(S_t\left(1-\mu+\frac{\sigma}{\xi}\right)-K\right)\psi_2\right]
\end{aligned}
\tag{18}
$$

The integral ψ_1 in (18) above can be solved by applying the change of variable $t = y^{-1/\xi}$, and then it can be evaluated in terms of the incomplete Gamma function, yielding the following solution:

$$
\psi_1 = \int_H^0 y^{-1/\xi}\exp\left(-y^{-1/\xi}\right)dy = -\xi\,\Gamma\left(1-\xi, H^{-1/\xi}\right) \tag{19}
$$

The solution of integral ψ_2 in (18) is:

$$
\psi_2 = \int_H^0 \left(y^{-1-1/\xi}\right)\exp\left(-y^{-1/\xi}\right)dy = \left[\xi\exp\left(-y^{-1/\xi}\right)\right]_H^0 = \xi\left(-\exp\left(-H^{-1/\xi}\right)\right) \tag{20}
$$

Combining results for ψ_1 and ψ_2, we obtain a closed form for the GEV call option price:

$$
C_t(K) = e^{-r(T-t)}\left\{\frac{-S_t\sigma}{\xi}\Gamma\left(1-\xi, H^{-1/\xi}\right)-\left(S_t\left(1-\mu+\frac{\sigma}{\xi}\right)-K\right)\left(-e^{-H^{-1/\xi}}\right)\right\} \tag{21}
$$

Grouping the terms with S_t together we have:

$$
C_t(K) = e^{-r(T-t)}\left\{S_t\left(\left(1-\mu+\sigma/\xi\right)e^{-H^{-1/\xi}}-\frac{\sigma}{\xi}\Gamma\left(1-\xi, H^{-1/\xi}\right)\right)-K\,e^{-H^{-1/\xi}}\right\} \tag{22}
$$

Theoretically, the application of the Girsanov Theorem (see, Neftci [2000]) to option pricing implies that the empirical distribution and the risk neutral distribution need to have the same support. By the Girsanov Theorem, the price levels and the size of returns that are *Q*-impossible due to the application of the truncation condition in (11) should not be *P*-possible, and vice versa, in terms of the realized historical prices and returns. We find that the conditions of the Girsanov Theorem are satisfied for the sample period for which the implied GEV based RND is extracted from option prices. The analysis of this is given in Appendix D.

Following similar steps, we can also derive a closed form solution for the put option price under GEV returns.[10] Details of this derivation can be found in the Appendix A, which yields the following equation:

[10] Once the call pricing formula is derived, one could simply obtain the put pricing formula using the put-call parity relationship. We numerically verified that the independently derived call and put pricing formulas satisfy put-call parity.

$$P_t(K)=e^{-r(T-t)}\left\{K\left(e^{-h^{-1/\xi}}-e^{-H^{-1/\xi}}\right)-S_t\left(\left(1-\mu+\sigma/\xi\right)\left(e^{-H^{-1/\xi}}-e^{-h^{-1/\xi}}\right)-\frac{\sigma}{\xi}\Gamma\left(1-\xi,h^{-1/\xi},H^{-1/\xi}\right)\right)\right\} \qquad (23)$$

where $h=1+\xi/\sigma(1-\mu)>0$. Note that *h* is a constant, given a set of parameters μ, σ, and ξ. In the following sections, we will analyse the properties of the GEV RND based closed form solutions for the call and put options under different moneyness conditions and values for the tail shape parameter.

Analysis of the GEV call option pricing model

This section aims to give some insights into the closed form solution for the GEV based call option pricing equation given in (22), which has two components and respective probability weights involving S_t and *K*. These two components can be interpreted along the same lines as the Black-Scholes model. The key to understanding the GEV option pricing formula lies with the term,

$$e^{-H^{-1/\xi}}=e^{-\left(1+\frac{\xi}{\sigma}\left(1-\frac{K}{S_t}-\mu\right)\right)^{-1/\xi}} \qquad (24)$$

This term is the cumulative GEV distribution function as defined in (2.a) for the “standardized moneyness” of the option defined as $(S_t - K)/S_t$. Hence, it corresponds to the risk neutral probability *p* of the call option being in the money at maturity.[11] For a given set of implied GEV parameters $\{\mu, \sigma, \xi\}$ we can work out (see, Exhibit 4) the range of exercise prices *K* in relation to the given S_t which yield**:** $e^{-H^{-1/\xi}}$ = 1 for deep in-the-money call options, $e^{-H^{-1/\xi}}$ = 0 for deep out-of-the-money call options, and $0 < e^{-H^{-1/\xi}} < 1$ for all other cases.

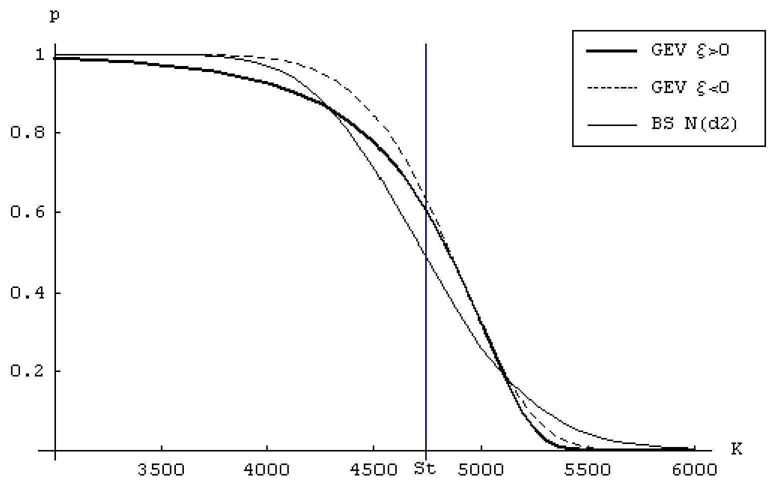
Exhibit 4 below plots the probability $p = e^{-H^{-1/\xi}}$ of exercising the option at maturity, given by the GEV model with two different values of ξ, and also for the Black-Scholes model.[12] When $\xi>0$, the density function of losses is Fréchet, and thus, the implied price density function is left skewed with a fat tail on the left, as shown respectively in Exhibits 2c and 3c. Since the latter implies there is a higher probability of downward moves of the underlying than in the Black-Scholes case, we see from Exhibit 4 how the probability of exercising the call option when $\xi > 0$ approaches 1 much slower than for the Black-Scholes model.

Exhibit 4

Probability of the call option being in the money at maturity

[11] Recall that in the case of the Black-Scholes model the probability of the option being in the money at maturity is given by $N(d_2)$, where N() is the standard cumulative normal distribution function, and $d_2=[\ln(S_t/K)+(r-\sigma^2/2)T]/\sigma\sqrt{T}$

[12] To make the three cases comparable, we use the same traded call option price data to estimate the GEV model and the Black-Scholes model. Then, to obtain the second case for the GEV model, we fix ξ to be equal to the initial estimate, but with opposite sign, and estimate the other two GEV parameters.



Notes: The probability of the call option being in the money at maturity is given by $p^{GEV}=exp(-H^{-1/\xi})$ for the GEV case, and by $p^{B\text{-}S}=N(d_2)$ for the Black-Scholes. The positive and negative values of ξ used for the GEV distribution are 0.16 and -0.16 respectively.

On the other hand, when $\xi < 0$, the GEV density of the losses is of Weibull type, and thus the implied price density function is right-skewed, resulting in a higher probability of upward moves. Therefore, the probability *p* of exercising the option as we lower the strike price *K* reaches 1 faster than in the Black-Scholes case. Note that for high strike prices and for any value of $\xi \neq 0$, the probability of the option being in the money goes to zero faster than for the Black-Scholes case.

When the call option is deep in-the-money (ITM) with $K << S_t$ and $e^{-H^{-1/\xi}} = 1$, the call price converges to a linear function of the expected payoff (see Appendix C for proof). Thus,

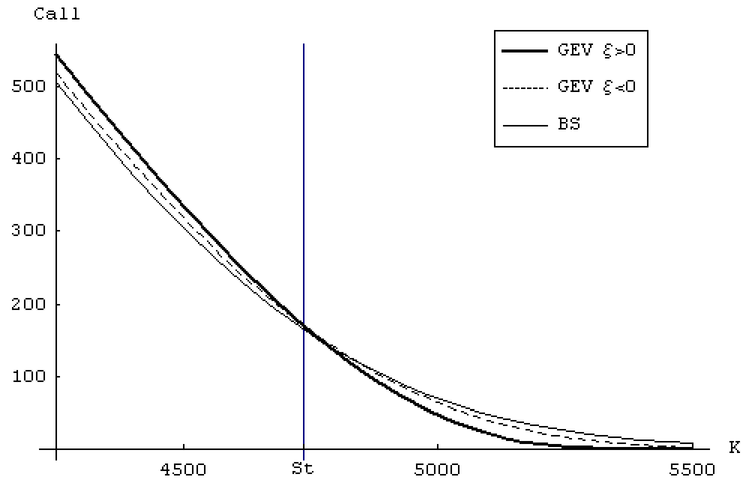
$$C_t(K) = e^{-r(T-t)}\left(E^Q_{t\,GEV}\left[S_T\right] - K\right) = S_t - e^{-r(T-t)}K \tag{25}$$

Here $E^Q_{t\,GEV}\left[S_T\right]$ is the conditional first moment of the price RND function, which by the martingale condition in (6) equals S_t. For this range of strike prices, the option prices obtained with the GEV model converge to those given by the Black-Scholes model. When the option is deep out of the money, then $K >> S_t$ and $e^{-H^{-1/\xi}} = 0$, and it is easy to verify that the call price is zero.

Exhibit 5 displays the call option prices obtained from the GEV model and the Black-Scholes model. The Black-Scholes model overprices the out-of-the-money (OTM) call options relative to the GEV model in both cases. In the OTM case, the GEV model yields higher values of call prices when $\xi< 0$ than when $\xi > 0$. This is because when $\xi< 0$, upward movements in the underlying price are more likely and the price density is truncated on the left (see Exhibit 3a). On the other hand, when $\xi > 0$ downward movements in the price are more likely and the price density function is truncated on the right (see Exhibit 3c). Hence, in the OTM region of high exercise prices, *K*, the GEV price with $\xi > 0$ gives the lowest prices for the call option.

Exhibit 5

Call option prices for the GEV model and the Black-Scholes model



Notes: The positive and negative values of ξ used for the GEV distribution are 0.16 and -0.16 respectively.

For in-the-money (ITM) options, as seen in Exhibit 5, the Black-Scholes model under prices call options when compared to the GEV model, and the GEV model gives higher option prices when $\xi > 0$ than when $\xi < 0$. This can be explained in terms of the asymmetry in the peakedness of the two densities. When $\xi > 0$, the RND function for the price is left skewed, with peakedness at higher values of the underlying than when $\xi < 0$. For at-the-money (ATM) options, the prices given by both models are approximately the same. Note that for deep ITM options, i.e. for much lower values of *K* (not shown in the graph) both GEV and Black-Scholes prices converge to the present discounted value of the intrinsic value of the option, increasing linearly as *K* falls.

Analysis of the GEV put option pricing model

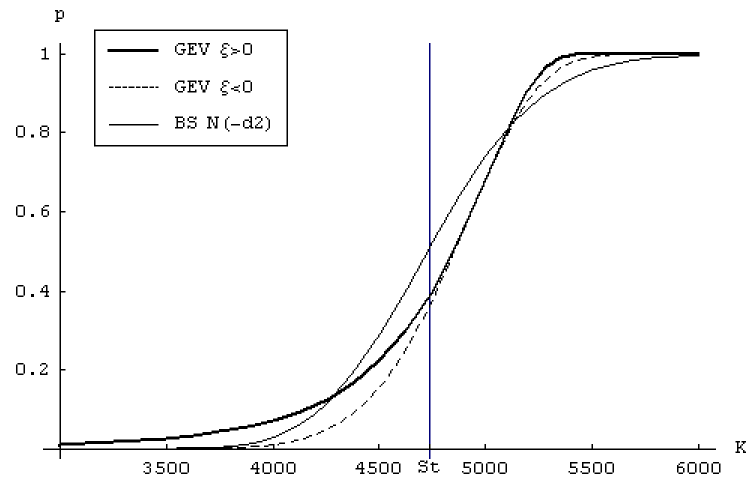
The analysis for the closed form solution of the GEV put option pricing model in equation (23) is analogous to what was done in the case of the call option. The probability of a put being in the money is given by

$$e^{-h^{-1/\xi}} - e^{-H^{-1/\xi}} \approx 1 - e^{-H^{-1/\xi}} \tag{26}$$

Here, note $e^{-h^{-1/\xi}}$ is approximately equal to 1 and hence (26) is one minus the probability of the call being in the money at maturity. In Exhibit 6, while considering the case of a Fréchet distribution for losses with $\xi > 0$, for low strike prices relative to the underlying, we have a greater probability of the put option being in the money at maturity as compared to either the Black-Scholes case or the GEV case when $\xi < 0$.

Exhibit 6

Probability of the put option being in the money at maturity

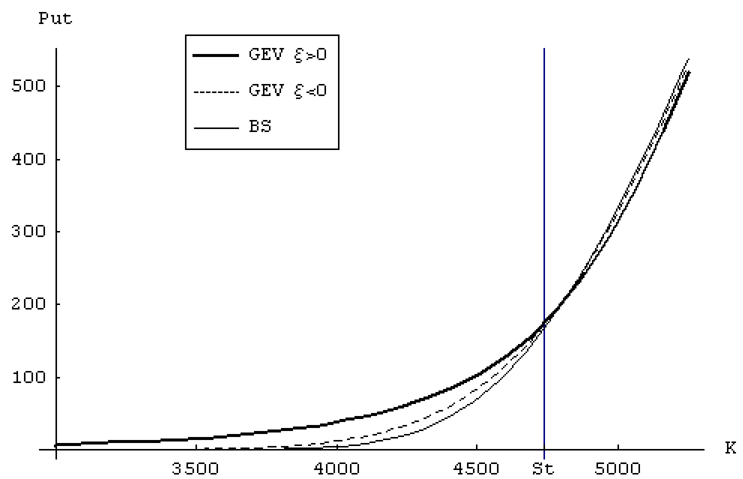


Notes: the probability of being in the money for the put option at maturity is given by $p^{GEV}=1 - \exp(-H^{-1/\xi})$ for the GEV case, and by $p^{B\text{-}S}=N(-d_2)$ for the Black-Scholes. The positive and negative values of ξ used for the GEV distribution are 0.16 and -0.16 respectively.

Exhibit 7 below displays the put option prices obtained with the GEV model along with the Black-Scholes model. The Black-Scholes model substantially underprices the out-of-the-money (OTM) put options relative to the GEV model. The GEV model yields higher values of OTM put prices when $\xi > 0$ than when $\xi < 0$. For in-the-money (ITM) put options, the Black-Scholes model only marginally overprices put options with respect to the GEV model. The GEV model gives higher prices for ITM put options when $\xi < 0$ than when $\xi > 0$. For at-the-money (ATM) options, the prices given by both the GEV and the Black-Scholes models are approximately the same. Note that for deep ITM put options, both GEV and Black-Scholes prices converge to the present discounted value of the intrinsic value of the option, $e^{-r(T-t)}K - S_t$, which increases linearly with *K*.

Exhibit 7

Put option prices for the GEV model and the Black-Scholes model



Notes: The positive and negative values of ξ used in the case of the GEV model are 0.16 and -0.16 respectively.

RESULTS

Data description

The data used in this study are the daily settlement prices of the FTSE 100 index call and put options published by the London International Financial Futures and Options Exchange (LIFFE). These settlement prices are based on quotes and transactions during the day and are used to mark options and futures positions to market. Options are listed at expiry dates for the nearest four months and for the nearest June and December. FTSE 100 options expire on the third Friday of the expiry month. The FTSE 100 option strikes are in intervals of 50 or 100 points depending on time-to-expiry, and the minimum tick size is 0.5. There are four FTSE 100 futures contracts a year, expiring on the third Friday of March, June, September and December.

The LIFFE exchange quotes settlement prices for a wide range of options, even though some of them may have not been traded on a given day. In this study we only consider prices of traded options, that is, options that have a non-zero traded volume on a given day. The data was also filtered to exclude days when the cross-section of options had less than three option strikes. Also, options whose prices were quoted as zero, had less than one week to expiry, or more than 120 days to expiry were eliminated. Finally, option prices were checked for violations of the monotonicity condition.[13]

The period of study is from 2-Jan-1997 to 1-Jun-2009. Exhibit 8 below summarizes the average number of traded option prices available on a daily basis, across both strikes and maturities, for each of the years in the period under study, including both call and put options. We can see how the number of traded contracts has increased substantially through time, from an average of 45 daily traded option prices in 1997 to 159 such prices in 2009. The range of strikes with options traded has also widened through time.

Exhibit 8
Summary data on FTSE-100 Index option prices

Period	Number of option prices (daily average)	Minimum strike	Maximum strike
1997	45	3125	5625
1998	61	3625	6825
1999	82	4025	7625
2000	98	4025	9125
2001	127	3775	7125
2002	126	2125	6725
2003	118	2425	5825
2004	124	2625	5825
2005	112	2325	5825
2006	114	2525	6625
2007	144	4025	7725

[13] Monotonicity requires that the call (put) prices are strictly decreasing (increasing) with respect to the exercise price. A small number of option prices that did not satisfy this condition were removed from the sample.

2008	164	3625	9025
2009	159	2600	9025
All Years	**113**	2125	9125

Notes: Average number of traded option prices available per day, minimum and maximum strike price with options trade per year (Jan 1997-June2009).

The European-style FTSE100 options, though they are options on the FTSE 100 index, can be considered as options on FTSE-100 index futures, because the futures contract expires on the same date as the option. Therefore, the futures will have the same value as the index at maturity, and can be used as a proxy of the underlying FTSE 100 index. By using this method, we avoid having to use the dividend yield of the FTSE 100 index, and the martingale condition in (6) becomes:

$$F_t = E_t^Q(S_T) \tag{27}$$

Here F_t is the price of the FTSE 100 futures contract at t, and S_T is the FTSE 100 index at maturity T. This martingale condition can be used to reduce the number of parameters in the GEV model from 3 to 2. This is analogous to the procedure in the Black-Scholes model where the mean of the distribution is obtained from the martingale condition and only the volatility parameter needs to be estimated. In the GEV case, the mean of the distribution does not directly correspond to the location parameter. Instead, the mean of the GEV distribution, as shown in Reiss and Thomas [2001], is a function of all three parameters and is defined as $\mu + (\Gamma(1-\xi)-1)/\xi)\sigma$. We can use this definition of the GEV mean together with (6) to express the location parameter μ in terms of the futures price $F_{t,T}$, current spot price, and the GEV scale and tail shape parameters σ and ξ:

$$\mu = 1 - \frac{F_{t,T}}{S_t} - \left[\frac{\Gamma(1-\xi)-1}{\xi}\right]\sigma \tag{28}$$

The risk-free rates used are the British Bankers Association’s 11 a.m. fixings of the 3-month Short Sterling London InterBank Offer Rate (LIBOR) obtained from the website www.bba.org.uk. Even though the 3-month LIBOR market does not provide a maturity-matched interest rate, it has the advantages of liquidity and of approximating the actual market borrowing and lending rates faced by option market participants (Bliss and Panigirtzoglou [2004]).

The option data used in this study can be divided into 6 moneyness categories given in Figlewski [2002].[14] Exhibit 9 below reports the number of observations for call and put options in each category of moneyness and maturity.

Exhibit 9
Number of observations in each maturity and moneyness category

[14] Figlewski [2002] explains that a measure of option moneyness should include an adjustment for volatility and maturity. Following his definition, we calculate the moneyness of an option as $\ln(S_t / Ke^{-rT})/(\sigma^{BS}\sqrt{T})$, where σ^{BS} is the Black-Scholes implied volatility for each option.

	Number of Observations	
Subsample	Calls	Puts
Maturity		
< 30 days	25,710	31,672
30 to 60 days	24,553	31,534
60 to 90 days	15,463	19,597
90 to 120 days	8,159	9,905
Moneyness		
deep OTM	8,421	17,895
OTM	23,507	38,043
ATM	30,086	26,808
ITM	8,171	5,790
deep ITM	3,701	4,172
Total	73,885	92,708

Notes: Number of observations of traded options for different maturity and moneyness categories (January 1997- June 2009).

Here, moneyness for a given option indicates how many standard deviations, σ, the strike price is away from the current underlying price in terms of the volatility, maturity of the option. An option is deep *out-of-the-money* (Deep OTM) if it is more than 1.5 σ out of the money, and similarly, it is deep in-the-money (Deep ITM) if it is more than 1.5 σ in-the-money. An option is classified as being at the money (ATM) if it is 0.5 σ in either direction of OTM and ITM. An additional classification is done for maturity, in terms of days to expiration. Note that there are options data available for time to expiration longer than 120 days, but the number of prices available for such long time horizons is small and the options are traded less frequently.

As can be seen in Exhibit 9, the short to medium term time to maturity, the first two groups, have the greatest number of data points, for both puts and calls. In terms of moneyness, the largest number of observations is found in the OTM and ATM category, while the deep ITM has the least number of observations for both puts and calls (ITM options are typically very expensive, as the option premium includes the intrinsic value, and thus are not traded often).

Empirical Methodology

For each quarterly expiration date in our data period, a total of 49 from March 1997 to March 2009, a target observation date was determined with horizons of 90, 60, 30 and 10 days to maturity. If no options were traded on the target observation date, the nearest date with traded options was used. All traded option prices available for each target observation date were used, subject to the filters discussed above, across all strikes and across all maturities, giving a one year constant horizon implied RND.[15] The implied RND was derived using the GEV and the Black-Scholes option pricing models.

[15] In order to estimate a single scale parameter to fit the prices of options across multiple horizons, we need to annualise the scale parameter σ in the pricing equations. This is similar to the procedure used in the Black-Scholes model such that the implied volatility parameter represents an annualised value. To achieve this, at the estimation stage, we replace the GEV scale parameter sigma σ by $\sigma\sqrt{T}$, where *T* is the time to maturity of the option, in number of years.

For each of these target observation dates a single implied RND was fitted using both put and call prices.

The GEV model was estimated by minimizing the sum of squared errors (SSE) between the option prices $\tilde{D}$ given by the analytical solution of the GEV option pricing equations in (22) and (23) and the observed traded option prices D (including both calls and puts) with strikes K_i, as indicated below:

$$SSE(t) = \min_{\zeta,\sigma}\left\{\sum_{i=1}^{N}\left(D_t(K_i) - \tilde{D}_t(K_i)\right)^2\right\} \tag{29}$$

Note that for the GEV model, we minimise the sum of squared errors with respect to only two GEV parameters, ie. scale and tail shape parameters, σ and ξ. We use equation (28) to substitute out the location parameter μ which has been derived as function of the futures price, $F_{t,T}$, current spot price, and these two parameters. For the Black-Scholes model, we likewise derive a single implied volatility parameter using both call and put prices. The optimization was performed using the non-linear least squares algorithm from the Optimization toolbox in MatLab.

In sample pricing performance

The in sample pricing performance tests consist of estimating the implied densities at time t, by using option prices at time t as well, and then analysing how well the model fits the same option prices. The pricing performance is reported in terms of the root mean square error RMSE, which represents the average pricing error in pence per option:

$$RMSE(t) = \sqrt{\frac{SSE(t)}{N}} \tag{30}$$

For each maturity horizon (ie. 10, 30, 60, 90 days) the average pricing error is taken over a total of 49 quarterly target observation dates from March 1997 to March 2009. The analysis that follows in Exhibit 10 reports the average pricing errors in terms of RMSE for each of these horizons used, to highlight some of the interesting pricing biases that are observed.

The GEV option pricing model outperforms the Black-Scholes model for all time horizons, and for both puts and calls. In particular, the GEV model removes the large pricing bias that the Black-Scholes model exhibits for options far from maturity. For a 90 day horizon, the Black-Scholes model has an average pricing error of 20.71 pence, while the error for the GEV is almost three times smaller, at 7.44 pence. Both models display an improvement in performance as time to maturity decreases. For close to maturity options, at a 10 day horizon, the GEV model continues to produce lower pricing errors, although the difference between the two models becomes smaller. Another observation is that the Black-Scholes model exhibits larger pricing errors for puts than for calls. In contrast, the GEV model exhibits similar sized errors for puts and call contracts. As anticipated from discussion on the

GEV put option result, it is important to note that for put options, the Black-Scholes model suffers a far greater deterioration in pricing performance when compared to the GEV model. While on average, across all maturity days, the difference the errors for calls and puts for the GEV model is only 0.12 pence, for the Black-Scholes model, that difference is substantially larger at 2.36 pence, mostly driven by the differences between put and call errors for far from maturity options.

Exhibit 10
In-sample pricing performance

	90 days		60 days		30 days		10 days		All days	
	BS	GEV	BS	GEV	BS	GEV	BS	GEV	BS	GEV
Calls	17.73	7.16	14.60	4.38	8.87	4.10	6.62	5.82	**11.96**	**5.36**
Puts	22.78	7.46	17.58	4.85	10.30	3.96	6.67	5.64	**14.33**	**5.48**
All	20.71	7.44	16.30	4.66	9.79	4.06	6.67	5.72	**13.37**	**5.47**

Notes: In-sample pricing performance of the Black-Scholes (BS) and the GEV models, in terms of Average Root Mean Square Error for option prices in pence, for options with horizons of 90, 60, 30 and 10 days to maturity.

Analysis of the in-sample pricing bias

It has been well documented that the Black-Scholes model exhibits a pricing bias for out of the money and in the money options, while pricing more accurately at the money options (Rubinstein, 1985). The pricing bias is defined in equation (31) as the deviation of the model estimated price with respect to the observed market price for each option contract:

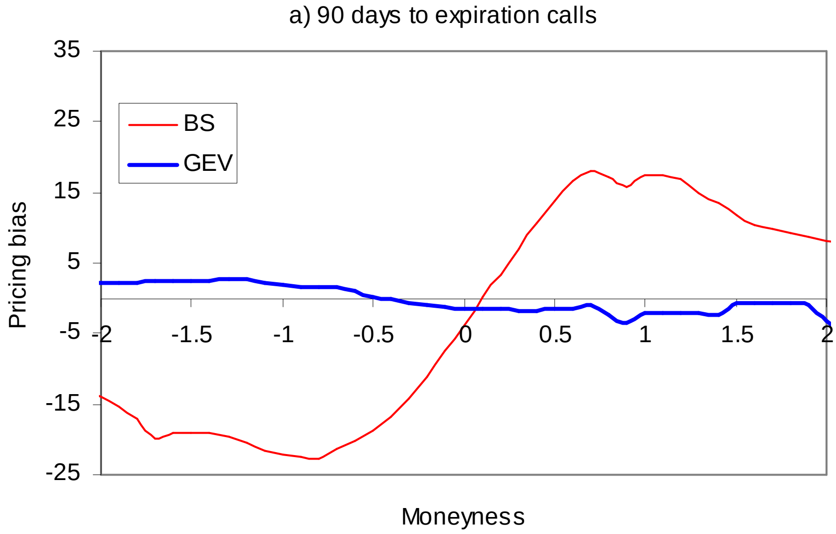
$$\textit{Price bias} = \textit{Market price} - \textit{Estimated price} \quad (31)$$

Here we take the individual option pricing errors obtained from the estimation done in the empirical methodology section, and report the average price bias across moneyness levels using a spline method.[16] The average pricing bias for call options is plotted below in Exhibit 11 for 90 and 10 day time horizon. For the 90 day time horizon, and in keeping with the results obtained in the previous section, the Black-Scholes model shows more deterioration in pricing accuracy for far from maturity contracts than for close to maturity ones. At far from maturity, the Black-Scholes model underprices ITM call options (moneyness from +0.5 to +1.5) by over 15 pence, while it overprices OTM call options (moneyness from -0.5 to -1.5) by around 20 pence. On the other hand, the price bias for the GEV model appears to be much less dependent on the moneyness levels, delivering much lower price bias across all moneyness levels. For the 10 day time horizon, we can see that for close to maturity call options, the Black-Scholes model exhibits the same pattern of price bias as for far from maturity options, but the magnitude of these price biases is much smaller. In line with results in the previous section, both models display a reduction in pricing bias as time to maturity decreases, and exhibit similar pricing biases oscillating between around +6 pence and – 6 pence.

[16] Given that at different target observation dates, we have different moneyness levels, we fit a spline to the pricing error observations as a function of moneyness on each day, and take the average of these splines across the 49 target observation dates for each horizon. Note we do not show price biases outside the [-2,+2] moneyness range as there are usually too few data points to obtain meaningful averages, but model prices tend to converge to market prices in the limits, either collapsing to 0 for very deep OTM or equalling the intrinsic value for very deep ITM.

Exhibit 11

Average call price bias in terms of moneyness



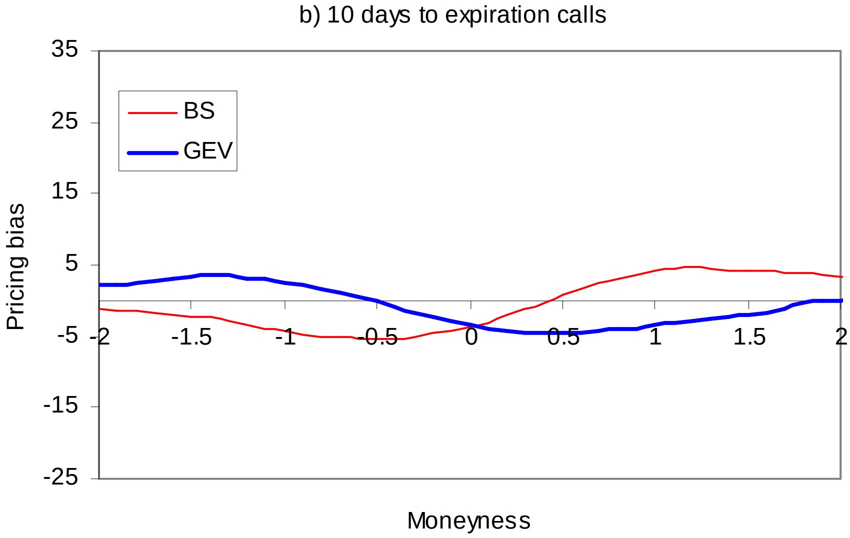
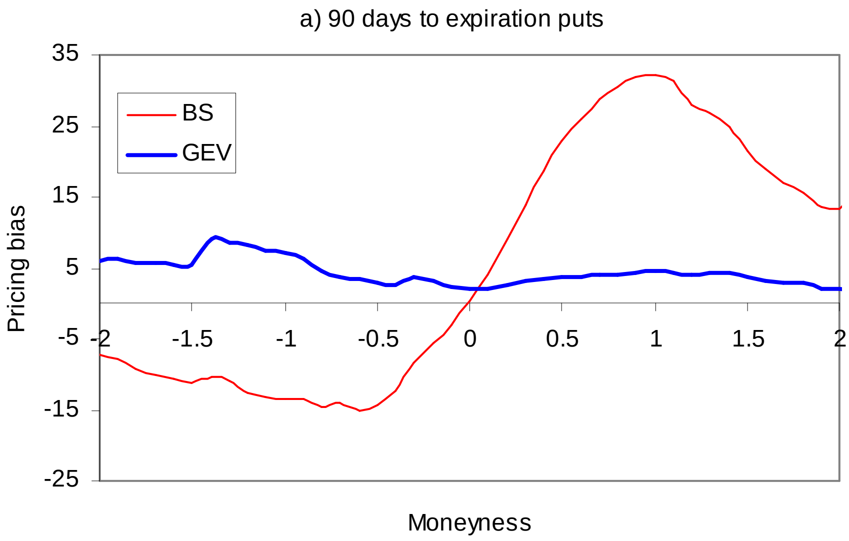
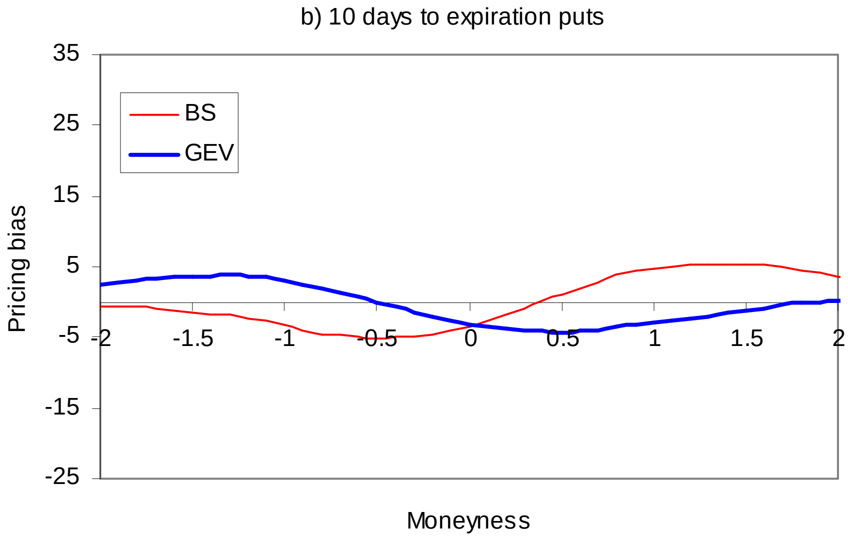


Exhibit 12 below display the pricing bias for put options. For far from maturity put options, at 90 days to maturity, the Black-Scholes model overprices ITM put options (moneyness from -0.5 to -1.5) by over 10 pence, while underprices OTM put options (moneyness from +0.5 to +1.5) by up to 30 pence. On the other hand, the GEV model exhibits a small pricing bias across the board. For close to maturity options, the chart for a 10 day time horizon shows how both models exhibit price biases of similar magnitude of around ±6 pence.

Exhibit 12

Average price bias for puts in terms of moneyness





OUT OF SAMPLE PRICING PERFORMANCE

For testing the out of sample pricing performance of the models, we calculate the model based option prices for contracts traded at *t+1*, with parameters that were estimated at time *t* in empirical methodology section, where *t* are the target observation dates described above. Then, we calculate the pricing errors as the differences between the forecasted option prices at time *t* and the market option prices known at time *t+1*.

These out of sample pricing errors are shown in Exhibit 13 below, in terms of RMSEs. They follow a similar pattern to the one reported for the in sample pricing results. In all cases, the GEV delivers smaller pricing errors than Black-Scholes. There is clearly some deterioration in the out of sample pricing performance for both the Black-Scholes and the GEV option pricing models. Again, the GEV model posts uniformly good performance over all the maturity periods while Black-Scholes does not and further the GEV model seems to be best placed to price 30 day maturity options. The Black-Scholes model seems to have weakness pricing put options while the GEV model has a marked advantage in this area.

Exhibit 13
Out of sample pricing performance

	90 days		60 days		30 days		10 days		All days	
	BS	GEV	BS	GEV	BS	GEV	BS	GEV	BS	GEV
Calls	17.86	8.64	16.12	7.68	10.32	6.45	7.54	6.27	**12.96**	**7.26**
Puts	22.78	8.53	17.76	7.87	10.33	6.08	7.73	5.83	**14.65**	**7.08**
All	21.03	8.66	17.49	7.94	10.50	6.29	7.68	6.01	**14.17**	**7.22**

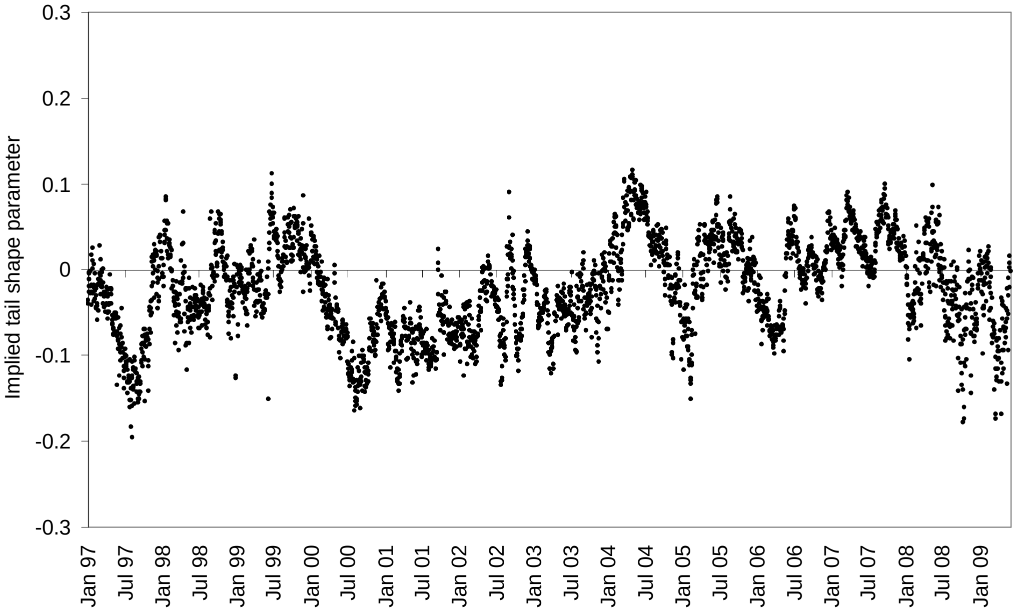
Notes: One day ahead pricing bias of the Black-Scholes (BS) and the GEV models, in terms of RMSE for option prices in pence, for options with horizons of 90, 60, 30 and 10 days to maturity.

The implied tail shape parameter

The time series of the implied GEV tail shape parameter, ξ, from 2-Jan-1997 to 1-Jun-2009 is displayed in Exhibit 14. These were obtained by estimating a single implied GEV density for all options selected by our filter for every day of the sample.[17] Note the implied tail index values are obtained for a constant maturity horizon and hence they do not suffer from maturity effects. The median standard error of these ξ estimates is 0.0077, thus, resulting in the majority of these tail shape estimates being significantly different than zero. We see that the implied tail index exhibits time variation, switching between negative values, which imply a finite tailed Weibull distribution, and positive values, which imply a fat tailed Fréchet distribution. It is important to note how the GEV distribution is flexible to capture shifts in tail movements with fat tailed behaviour being interspersed with more normal market conditions. For example, from 2000 to July 2002, there was a period of relative calm while periods of severe market falls or crisis coincide with a Fréchet type implied GEV distribution characterized by a positive tail shape, such as during LTCM crisis in 1998, and the credit crisis in 2007-8. The next section will look at some of these crisis periods in more detail.

Exhibit 14
Time series of the implied GEV tail shape parameter ξ

[17] Here, we follow the same constant horizon methodology outlined earlier. Instead of only estimating the GEV implied density for some target observation dates, we estimate the GEV implied density for every trading day in the sample, in order to obtain a daily time series of the implied tail index.



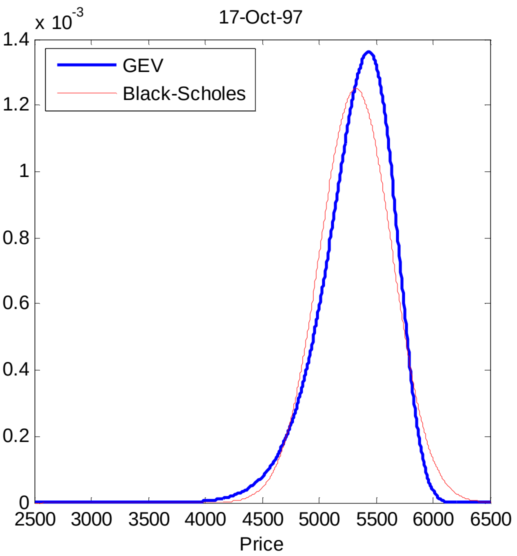
Event studies

In this section we compare the implied GEV distribution of prices before and after special events. Some of the major events that occurred within the period of study (1997 – 2009) are the Asian Crisis, the LTCM crisis, 9/11, and the collapse of Lehman Brothers during the 2007-8 credit crisis.

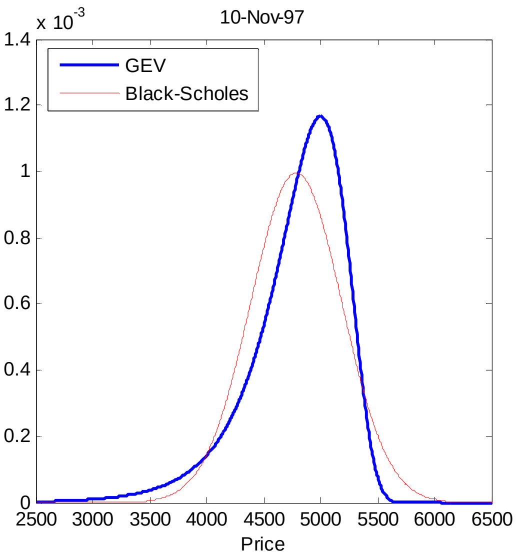
The Asian Crisis

Starting in July 1997, several major Asian markets experienced a downwards spiral of panic selling and price discounting, triggered by the currency devaluation in Thailand. These events, known as the Asian crisis, have been pinpointed to culminate around the 20th October 1997 (Gemmill and Saflekos, [2000]). Exhibit 15 below displays the implied RNDs for the GEV model and for the Black-Scholes model, one week before the Asian crisis on 14 October 1997 (left panel) and five weeks after it on 28 November 1997 (right panel). In both cases the GEV density exhibits negative skewness and a fatter than normal left tail, and they increase substantially after the Asian crisis, implying higher than normal probabilities of further downward moves. The tail shape parameter increased, going from a negative -0.09 value that implied a thin tailed Weibull distribution before the Asian crisis, to a positive 0.05, which implied a fat tailed Frechet type distribution afterwards. Implied kurtosis increased substantially, almost doubling after the event, implying that extreme losses become more probable. A detailed summary of implied moments is given at the end of this section.

Exhibit 15
Implied RND functions around the Asian crisis



a) Before Asian crisis

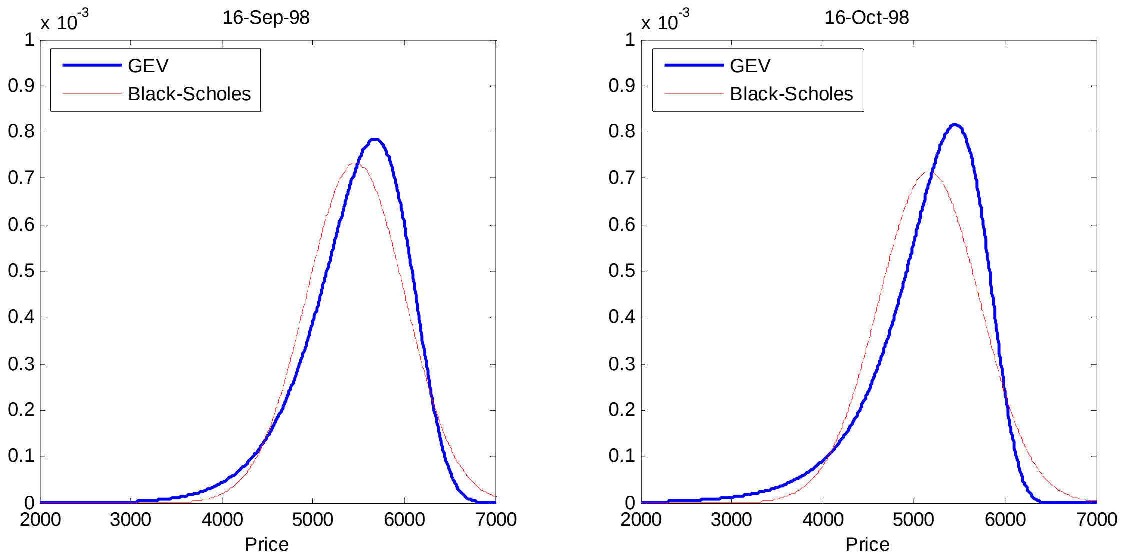


b) After Asian crisis

The LTCM Crisis

Long Term Capital Management (LTCM) was a hedge fund founded in 1994 by a group of renowned traders and academics, who raised \$1.3 billion at inception. The main strategy of the fund was convergence arbitrage, and during the first two years its returns were close to 40%. However, at the end of September 1998, the fund had lost substantial amounts and was close to default. To avoid the threat of a global systemic crisis, on 23 September 1998 the Federal Reserve organized a \$3.5 billion rescue package. A group of leading banks took over the management of the fund in exchange for 90% of its equity. Exhibit 16 below shows the implied RND functions one week before the major events in the LTCM crisis, on 16 September 1998 (left panel), and three weeks after, on 16 October 1998 (right panel). The tail shape parameter ξ increased from -0.06 before the LTCM crisis, to 0.03 after, with both implied skewness and kurtosis increasing substantially after the event. This indicates a greater degree of uncertainty in the market which is manifested in a high implied probability of further market downturns.

Exhibit 16
Implied RND functions around the LTCM crisis



a) Before LTCM crisis

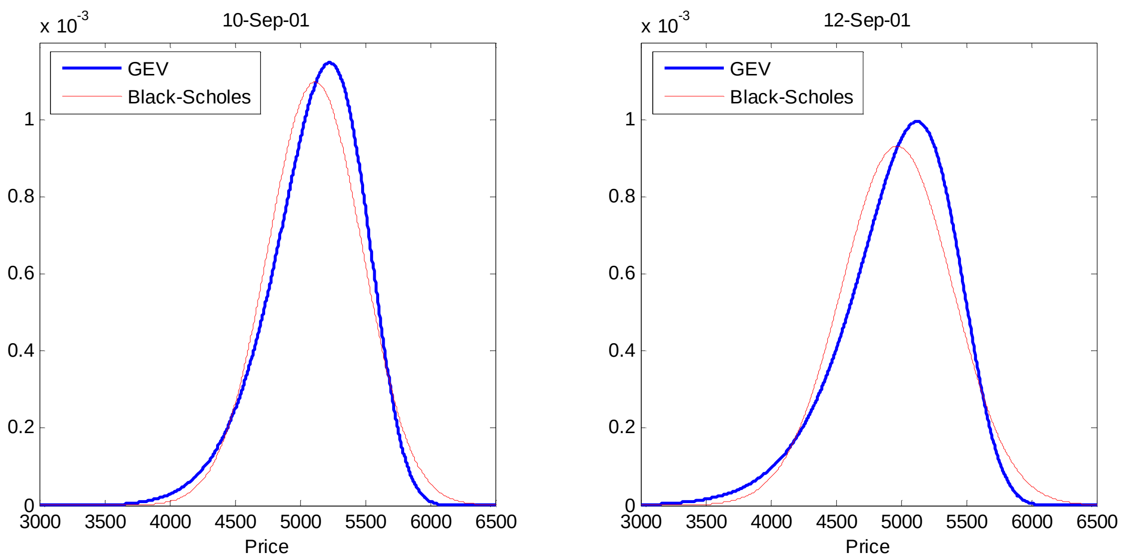
b) After LTCM crisis

<u>The 11 September 2001 Terrorist Attacks</u>

The 9/11 terrorist attacks caused a sudden drop in markets around the world, with the FTSE 100 suffering a loss of -2.6%, one of the worst daily loss in the period of 1996 to 2009. Investors feared the attacks would hasten a recession which was already looming for the UK. Exhibit 17 shows the implied RND functions for the day before the attacks, 10 September 2001, on the left panel, and the RNDs for a day after the event, 12 September 2001, on the right panel. The implied tail shape parameter ξ increased from -0.11 the day before the events, to 0.05 a day after the events, implying a fattening of the tail. This indicates the market expectations of further falls increased quite rapidly after this event, similar to what we saw in the previous two cases.

Exhibit 17

Implied RND functions around 9/11 event



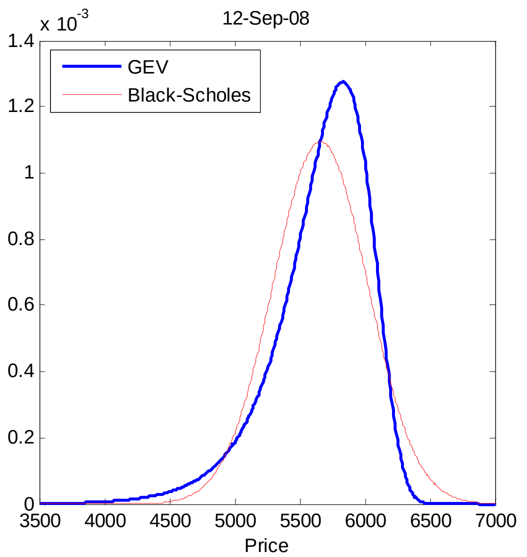
a) Before 9/11 events

b) After 9/11 events

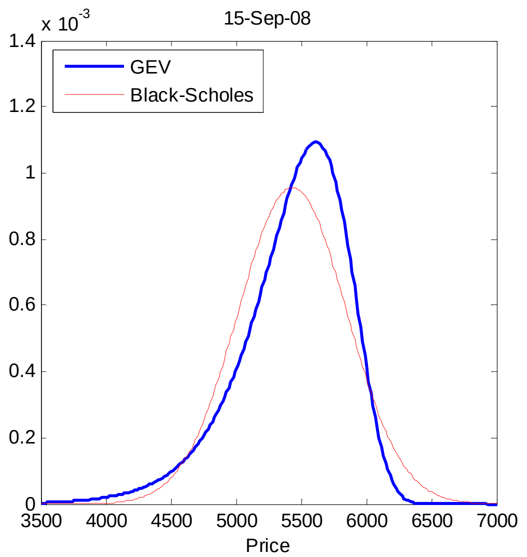
The collapse of Lehman Brothers

Lehman Brothers filed for Chapter 11 bankruptcy protection on Monday September 15, 2008. Exhibit 18 shows the implied RND functions extracted from option prices before the event, on Friday September 12, and on the day of the bankruptcy filing, Monday, September 15. We can see that the implied GEV distribution before the event was already exhibiting substantially negative skewness and a fat tail on the left, reflecting the negative sentiment in the market about the ongoing credit crisis which started in 2007. This negative sentiment was partially driven by the uncertainty about further write-downs and credit-worthiness of financial institutions, having already seen the collapse of Bear Stearns in March that year. We can see how the fat tail of the implied distribution became even more pronounced on Monday 15 September when Lehman Brothers filed for bankruptcy.

Exhibit 18
Implied RND functions around the collapse of Lehman Brothers



a) Before the collapse of Lehman Brothers



b) After the collapse of Lehman Brothers

We have seen how the non-Gaussian characteristics of implied RNDs became more accentuated after each of the four events, with the left tail of the implied RND becoming thicker, and the distribution more left skewed, further deviating from the Gaussian Black-Scholes implied distribution. In order to quantify these changes, Exhibit 19 below displays the GEV implied parameters before and after each of the four crisis events, and the implied moments of the GEV distribution, as well as the volatility implied by the Black-Scholes model. Without exception, the implied tail shape index, ξ, of the GEV distribution switches from negative or zero to positive after the crisis event. The implied volatility increases for both GEV and Black-Scholes models after the events, reflecting an increase of the uncertainty of the market. Indeed, the GEV and Black-Scholes implied volatilities are remarkably identical. However, the higher moments implied by the GEV model already differ from the ones of a normal distribution even before the events, and non-normal characteristics of the implied GEV RNDs further increase after the events, indicating a growing asymmetry in the distribution and greater probability of extreme losses. The fact that the GEV model is highly sensitive to changes in market sentiment and captures increased fear of further price falls is in line with previous studies on the use of non-Gaussian RND analysis such as Gemmill and Saflekos [2000]. It is beyond the scope of this article to establish whether implied GEV based RND functions can predict price falls rather than be useful in capturing the change in market sentiment after or coincidental with the crisis event.

Exhibit 19
Summary statistics around event studies

Event	Date	σ	ξ	Implied Volatility	Implied Skewness	Implied Kurtosis	Implied Vol BS
Asian crisis	17-Oct-97	0.18 (0.003)	-0.09 (0.008)	19.1%	-0.60	3.47	19.8%
	10-Nov-97	0.23 (0.006)	+0.05 (0.024)	27.5%	-1.38	6.72	26.4%
LTCM	16-Sep-98	0.31 (0.007)	-0.06 (0.016)	37.6%	-0.84	4.19	34.7%
	16-Oct-98	0.30 (0.005)	+0.03 (0.009)	39.9%	-1.30	6.23	37.7%
9/11	10-Sep-01	0.22 (0.002)	-0.11 (0.003)	24.3%	-0.60	3.48	24.8%
	12-Sep-01	0.27 (0.004)	-0.05 (0.011)	28.8%	-0.89	4.35	30.0%
Lehman Brothers	12-Sep-08	0.19 (0.022)	+0.00 (0.054)	22.4%	-0.85	4.20	22.5%
	15-Sep-08	0.24 (0.013)	-0.05 (0.020)	25.7%	-0.99	4.75	26.8%

Notes: GEV parameters with standard errors, implied moments by the GEV based RND functions, and implied Black-Scholes volatility around crisis events.

CONCLUSIONS

We have developed a new option pricing model that is based on the GEV distribution, and have obtained closed form solutions for the Harrison and Pliska [1981] no arbitrage equilibrium price for the European call and put options. It was argued that the GEV density function for negative asset returns, which in turn yielded the GEV based RND function, has great flexibility in defining the tail shape of the latter implied by traded option price data without *a priori* restrictions on the class of distributions. In particular, no *a priori* restriction is imposed that the GEV distribution function of negative returns belongs to the Fréchet class with fat tails. The traded option data is used to select the GEV RND function which displays left skewness and leptokurtosis for the underlying under the risk neutral measure. Some recent option pricing models that aim to capture the leptokurtosis and left skew in the RND function, in contrast, start with a specific fat tailed distribution. Other option pricing models that attempt to overcome the drawbacks of the Black-Scholes model fail to obtain closed form solutions or have far too many parameters.

The closed form solution for the GEV based call option pricing model has properties analogous to the Black-Scholes price equation, especially with regard to the probability of the option of being in or out of the money at maturity. In the GEV case, the latter is governed by the cumulative distribution for the GEV, which is defined by the implied GEV parameters. The implications for the probability mass in the tails of the GEV density function with switches in the tail shape parameter, ξ, is shown to challenge the traditional understanding of tail behaviour from symmetric Gumbel class of distributions where 99% of the rises and falls of value occur within limited volatility range and with no skew in the probability mass. The skew in the density function in the case of positive and negative tail shape values implies large one directional movements and truncated probability mass beyond a certain value in the other direction. In other words, a simultaneous existence of infinite tails in both directions is typically unlikely except for Gumbel class of distributions where, ofcourse, extreme moves in either direction have non-zero but negligible probability.

From the analysis, there is a very clear indication that $\xi > 0$ results in a smaller probability for a call option being in the money at maturity compared to the Black-Scholes case. In contrast, for the put option, $\xi > 0$ results in a higher probability of being in the money at lower strikes when compared to the Black-Scholes case. When applying the GEV option pricing model for the FTSE 100 index options, it was found that the GEV based in-sample pricing biases were substantially smaller than the ones from the Black-Scholes, for all times to maturity and at all moneyness levels. This improved pricing accuracy was also found in out-of-sample pricing tests, when forecasting one day ahead option prices with previous day's parameter estimates.

We showed how the implied tail shape parameter was found to be time varying, though stable enough to be useful in out of sample pricing. Cases of high positive ξ in the market implied density function, associated with the GEV-Fréchet class, usually coincided with periods of market falls and periods surrounding crisis events. For most other periods, the implied tail shape parameter indicated Weibull or Gumbel distributions.

In the event studies surrounding particular crisis events, typically the implied tail shape parameter ξ increases after the crisis event, which indicates that the implied GEV distributions reflect the market sentiment of increased fear of downward moves. There is a large and growing literature on traded option implied statistics for their capacity to incorporate market information and for forecasting volatility and market distress (see,Giamouridis and Skiadopoulis [2010], and Poon and Granger [2003]). Given the in sample and out of sample option pricing capabilities of the GEV model for 30 days and longer time to maturity, there is clear indication that GEV based RNDs can deliver good results in terms of capturing market expectations of market distress beyond the 30 day horizon. These results are in line with those in Peng, Markose and Alentorn [2010], who have found that the GEV RND based implied volatility outperforms VIX type model free implied volatility measures for forecasting realized volatility. However, further research is needed to establish more rigorously whether, as noted in previous studies (Gemmill and Saflekos [2000]), the implied RND functions have predictive power regarding downward market movements or can only reflect these moves coincidental with the market crisis.

Future work will analyse the hedging properties of the GEV based option pricing model, and its scope for delta hedging. Further, the option market implied tail index and the GEV based RND have useful and interesting applications in risk management. In Markose and Alentorn [2008] the GEV based implied RND is used to obtain the so called Extreme Economic Value at Risk (EE-VaR), in line with the proposal to use a quantile based application of the option implied RND in Ait-Sahalia and Lo [2000]. It has been argued that E-VaR , in contrast to historical data based VaR (sometimes called statistical or S-VaR) is a more general risk measure since it incorporates the market's evaluation of risk, the demand–supply effects, and the probabilities that correspond to extreme losses (Panigirtzoglou and Skiadopoulos [2004]). Markose and Alentorn [2008] find that the GEV RND based VaR obtained from traded options can deliver better VaR performance than conventional methods. The explicit tail shape parameter of the GEV based RND is cited as the main reason for the capacity of this new model to flexibly and rapidly respond to extreme market movements which to date has not been adequately dealt with by other option pricing models.

APPENDIX A

Derivation of the put option price equation for ξ > 0

The derivation of the closed form solution for the put option price equation is similar to the derivation for the call option price equation. When applying the change of variable defined in (13) to the put option price equation, after having substituted for the price RND function *g(S_T)* in (10), the upper limit of integration *K* in the put option equation becomes *H* as defined in (16), while the lower limit of integration in the put option equation becomes $h = 1 + \xi(1-\mu)/\sigma$. Using these new limits of integration we have:

$$P_t e^{r(T-t)} = \frac{1}{\xi}\left[\frac{-S_0\sigma}{\xi}\int_h^H y\left(y^{-1-1/\xi}\right)e^{\left(-y^{-1/\xi}\right)}\,dy - \left(K - S_t\left(1-\mu+\frac{\sigma}{\xi}\right)\right)\int_h^H \left(y^{-1-1/\xi}\right)e^{\left(-y^{-1/\xi}\right)}\,dy\right]$$

$$= \frac{1}{\xi}\left[\frac{-S_0\sigma}{\xi}\psi_1 - \left(K - S_t\left(1-\mu+\frac{\sigma}{\xi}\right)\right)\psi_2\right] \qquad \text{(A-1)}$$

Evaluating the first integral in (A-1) yields:

$$\psi_1 = \int_h^H \left(y^{-1-1/\xi}\right)e^{\left(-y^{-1/\xi}\right)}dy = \left[\xi e^{\left(-y^{-1/\xi}\right)}\right]_h^H = \xi\left(e^{\left(-H^{-1/\xi}\right)} - e^{\left(-h^{-1/\xi}\right)}\right) \qquad \text{(A-2)}$$

To solve the second integral in (A.1), consider the change of variable $t = y^{-1/\xi}$, and $y = t^{-\xi}$

$dy = -\xi t^{-1-\xi}dt$, which yields:

$$\psi_2 = \int_{h^{-/\xi}}^{H^{-/\xi}} t\,e^{-t}\,t^{-1-\xi}d\left(t^{-\xi}\right) = -\xi\int_{h^{-/\xi}}^{H^{-/\xi}} t^{-\xi}\,e^{-t}\,dt = -\xi\int_{h^{-/\xi}}^{H^{-/\xi}} t^{(1-\xi)-1}\,e^{-t}\,dt \qquad \text{(A-3)}$$

We can solve this integral directly by using the definition of the generalized Gamma function:

$$\Gamma(a,z_0) - \Gamma(a,z_1) = \Gamma(a,z_0,z_1) = \int_{z_0}^{z_1} t^{a-1}e^{-t}dt \qquad \text{(A-4)}$$

and we obtain the following result:

$$\psi_2 = \int_h^H y^{-1/\xi}\exp\left(-y^{-1/\xi}\right)dy = -\xi\,\Gamma\left(1-\xi, h^{-1/\xi}, H^{-1/\xi}\right) \qquad \text{(A-5)}$$

Combining results for ψ_1 and ψ_2 and rearranging, we obtain a closed form solution for the put option equation:

$$P_t(K) = e^{-r(T-t)}\left\{K\left(e^{-h^{-1/\xi}} - e^{-H^{-1/\xi}}\right) - S_t\left(\left(1-\mu+\sigma/\xi\right)\left(e^{-H^{-1/\xi}} - e^{-h^{-1/\xi}}\right) - \frac{\sigma}{\xi}\Gamma\left(1-\xi, h^{-1/\xi}, H^{-1/\xi}\right)\right)\right\}$$

(A-6)

APPENDIX B

Derivation of the call option price equation for ξ = 0

The derivation of the closed form solution of the call option price when ξ = 0 follows a similar procedure to the one when ξ ≠ 0. By assuming that negative returns are distributed following the standardised GEV distribution when ξ = 0 given in (3.b), and applying the formula in (9) we obtain the RND function of the underlying price $g(S_T)$ in terms of the standardized GEV density function given in equation (3.b) as follows:

$$g(S_T) = \frac{1}{S_t\sigma}\exp\left(-\frac{(L_T-\mu)}{\sigma}\right)\exp\left(-\exp\left(-\frac{(L_T-\mu)}{\sigma}\right)\right) \quad \text{(B-1)}$$

Substituting $g(S_T)$ in (B.1) into the call price equation in (4), we have:

$$C_t(K) = e^{-r(T-t)}\int_K^{\infty}(S_T-K)\frac{1}{S_t\sigma}\exp\left(-\frac{(L_T-\mu)}{\sigma}\right)\exp\left(-\exp\left(-\frac{(L_T-\mu)}{\sigma}\right)\right)dS_T \quad \text{(B-2)}$$

Consider the change of variable:

$$y = -\frac{(L_T-\mu)}{\sigma} = \frac{1}{\sigma}\left(\frac{S_T}{S_t}-1+\mu\right) \quad \text{(B-3)}$$

Under this change of variable, the underlying price S_T and dS_T can be written in terms of y as follows:

$$S_T = S_t(1-\mu+\sigma y) \qquad \text{and} \qquad dS_T = S_t\,\sigma\,dy \quad \text{(B-4)}$$

Also, the density function in (B-1) for the underlying price at maturity in terms of y becomes:

$$g(y) = \frac{1}{S_t\sigma}\exp(y-\exp(y)) \quad \text{(B-5)}$$

Note now that under the change of variable the lower limit of integration for the call option equation in (B.2) becomes:

$$H = \frac{1}{\sigma}\left(\frac{K}{S_t}-1+\mu\right) \quad \text{(B-6)}$$

while the upper limit of integration in (B-2) remains infinity. Substituting for S_T and dS_T as defined in (B-4) into (B-2), using the new limits of integration, and rearranging we have:

$$C_t e^{r(T-t)} = \int_H^{\infty} \left(S_t (1-\mu+\sigma y) - K \right) \exp(y-\exp(y))\, dy$$
$$= S_t\, \sigma \int_H^{\infty} y \exp(y-\exp(y)\, dy + \left(S_t (1-\mu) - K\right) \int_H^{\infty} \exp(y-\exp(y))\, dy$$
$$= S_t\, \sigma \psi_1 + \left(S_t (1-\mu) - K\right) \psi_2 \tag{B-7}$$

The integral ψ_1 in (B-7) above can be evaluated in terms of the incomplete gamma function, by applying the change of variable $t = exp(y)$ and integrating by parts, yielding the following solution:

$$\psi_1 = \int_H^{\infty} y \exp(y-\exp(y))dy = \left[-\exp(-\exp(y)) - \Gamma(0, \exp(y))\right]_H^{\infty}$$
$$= \exp(-\exp(H)) + \Gamma(0, \exp(H)) \tag{B-8}$$

The solution of integral ψ_2 in (B-7) is:

$$\psi_2 = \int_H^{\infty} \exp(y-\exp(y)\, dy = \left[-\exp\left(-\exp(y)\right)\right]_H^{\infty} = \exp(-\exp(H)) \tag{B-9}$$

Combining results for ψ_1 and ψ_2 and rearranging, we obtain a closed form for the GEV call option price:

$$C_t(K) = e^{-r(T-t)} \left[S_t \left((1-\mu+\sigma) e^{-e^H} + \sigma \Gamma(0, e^H) \right) - K\, e^{-e^H} \right] \tag{B-10}$$

Following a similar procedure, the closed form solution of the GEV model for the put option price when $\xi = 0$ is found to be:

$$P_t(K) = e^{-r(T-t)} \left[K \left(e^{-e^h} - e^{-e^H} \right) - S_t \left((1-\mu+\sigma) \left(e^{-e^h} - e^{-e^H} \right) + \sigma \Gamma(0, e^h, e^H) \right) \right] \tag{B-11}$$

APPENDIX C

Simplification for deep in the money options

The GEV option pricing equation in (22) can be simplified for the special case when the call option is deep in the money, i.e $S_t >> K$. In that case, the terms involving H can be simplified, and as $e^{-H^{-1/\xi}} \to 1$, it implies that $H^{-1/\xi} \to 0$. Substituting these approximations into the call option equation in (22) and rearranging, the call option price becomes a linear function of K:

$$C_t(K) = e^{-r(T-t)} \left\{ S_t \left(1 - \mu + \frac{\sigma}{\xi} \left(1 - \Gamma(1-\xi)\right) \right) - K \right\} \tag{C-1}$$

The mean of the standardized GEV negative returns, L_T, distribution is given by (see Dowd, 2002, p.273):

$$E_{GEV}^{Q}[L_T] = \mu - \frac{\sigma}{\xi}(1 - \Gamma(1-\xi)) \qquad \text{(C-2)}$$

Using the definition in (7) in terms of negative returns, $L_T = -R_t = 1 - S_T / S_t$, and applying the expectations operator:

$$E_{GEV}^{Q}[S_T] = S_t\left(1 - E_{GEV}^{Q}[L_T]\right) = S_t\left(1 - \mu + \frac{\sigma}{\xi}(1 - \Gamma(1-\xi))\right) \qquad \text{(C-3)}$$

To satisfy the martingale condition in (6), it is clear that the expression in the last bracket can be taken to be $e^{r(T-t)}$ or the GEV risk neutral compounded return. Thus, we can rewrite the simplified equation for the call option in (C-1) to yield the result given in (25):

$$C_t(K) = e^{-r(T-t)}\left(E_{GEV}^{Q}[S_T] - K\right) = S_t - e^{-r(T-t)}K \qquad \text{(C-4)}$$

APPENDIX D

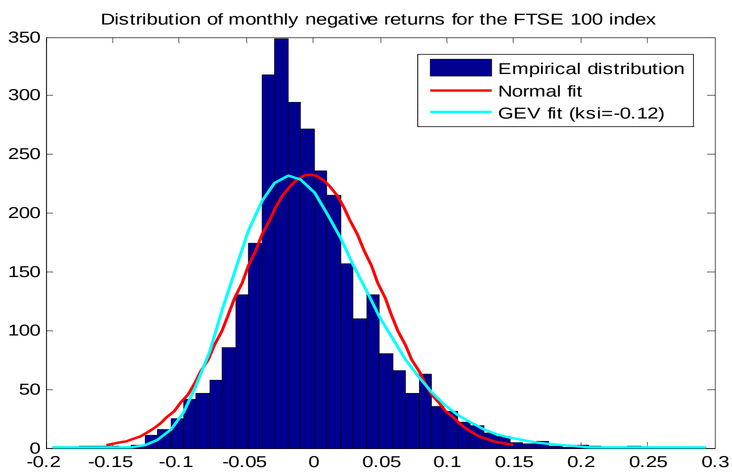
Empirical distribution of the FTSE100 index, GEV based RND and the Girsanov Theorem

Exhibit D1 below displays the empirical distribution, as a histogram, of rolling monthly negative returns for the FTSE100 index, from 1997 to 2009. We use monthly returns to represent a typical maturity horizon for the traded option implied RND method. We can clearly observe that a fat tail occurs for losses on the right hand side, while there is no evidence of fat tailedness for gains, on the left hand side.

Overlaid on top of the empirical distribution we have plotted the fitted normal distribution, as per the Black-Scholes model, as well as the empirical GEV distribution. The normal distribution function is only plotted between + 3 and -3 standard deviations, to indicate where over 99% of the probability mass is governed by the normal distribution. The exhibit shows that the empirical GEV distribution for negative losses has a long right tail and a truncated left one.

Exhibit D1

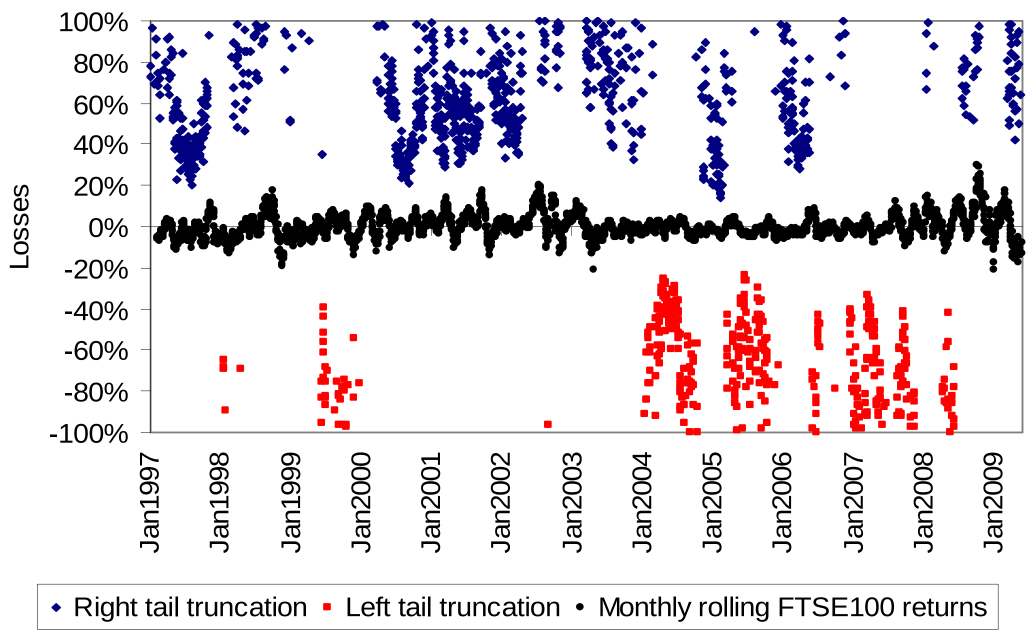
Empirical distribution of monthly negative returns for the FTSE 100 index



We need to verify that the Q-impossible events implied by the conditions for the GEV RND function in equation (11) are not P-possible.

Exhibit D2

Time series of truncation points, and realised rolling monthly FTSE 100 returns



We obtain the daily implied tail shape parameters, ξ, and the precise truncation values that are obtained in terms of the upper bound (maximum gain, left tail or gains) and lower bound (maximum loss, right tail for losses). This has been plotted in Exhibit D2 along with the realized negative returns which is centred on zero percentage returns. Note that less stringent truncation values for returns in excess of +/- 100% are not shown in the plot. The data for the upper and lower truncations values visible in the plot constitute only 45% of all possible days and on a large proportion of other days the Q-possible range of percentage gains and losses were well beyond even the +/- 100% band. As can be seen at no time does the realized returns violate the truncation points given by the implied tail shape parameter, ξ, for the GEV based RND.

ENDNOTES

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